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LETTER TO THE EDITOR

Approximate solution of the damped Burgers equation

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Abstract. To derive an approximate solution of the damped Burgers equation, we use the tanh method as a perturbation technique. As a result, a damped shock-wave structure appears which moves with a decreasing velocity. In particular a bump type of behaviour appears, after a certain time, in the tails of the solution.

When solving nonlinear wave equations, one usually looks for travelling waves so that one deals with ordinary differential equations. Several techniques are available to solve them: direct integration, transformation or substitution (in fact trial and error), or other more involved methods such as the Hirota method (1980), the Painlevé expansion (Kudryashov 1991) or the direct algebra method (Hereman and Tanaoka 1990). Unfortunately, most of them are rather complex and moreover, not adapted for use as a perturbation technique.

In 1990, Huibin and Kelin introduced a series expansion in terms of a tanh function, to solve a KdV–Burgers type of equation. A more systematic version of this tanh method was developed afterwards (Malfliet 1992, 1993) and applied to conservative systems. Some new results were established and the ease of use is remarkable. Like with other methods, stable waveforms are found which travel with a constant velocity.

In this letter we show that the tanh method also can be used to search for approximate solutions of nonlinear wave equations which arise in non-conservative systems. As an example we deal with a dissipative system, and in particular with the damped Burgers equation.

A perturbation theory to study those wave equations, starting from the exact solution which is known when dissipation vanishes, is not trivial to develop. Take for instance the KdV equation in which a linear damping is added. In that case a particular perturbation approach was used (see, for instance, Lamb (1980) and references therein). First, the time evolution of the scattering data, which originate from the inverse scattering technique, is investigated. Next, the dynamics of the relevant quantities is studied. The calculations, however, are cumbersome and despite considerable efforts, the results are not satisfactory. Moreover, such technique cannot be used in general and obviously not in this case.

We propose a somewhat different approach. In the first place, the unperturbed solution is taken as a starting point and we only allow a time-dependent amplitude and velocity. Next we introduce an infinite series expansion in tanh with time-dependent coefficients. In this way, calculations turn out to be straightforward. As an example we have chosen the (damped) Burgers equation. It appears as a model equation in fluid mechanics to describe diffusive waves (Sachdev 1987), subjected to dissipation. It is written as

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}+\lambda u=0. \tag{1}$$

We look for a travelling wave solution $u(x,t)$ of this nonlinear equation. If no dissipation is present, only one coordinate is required: $\xi=c(x-vt)$. Therefore, we introduce

$$u(x,t)=U(\xi,t) \qquad \text{with } \xi=c(x-\phi(t)). \tag{2}$$

The quantity $U(\xi,t)$ represents a (localized) solution, which travels with speed $\mathrm{d}\phi(t)/\mathrm{d}t$. It exemplifies a wave with a characteristic width $L=c^{-1}$, which plays the role of a wavelength (c: wavefactor). In contrast with other perturbation methods we do not allow any time dependence of that quantity c, in order to avoid secular terms (terms proportional to x).

After transformation from the (x,t) to the (ξ,t) variables, we get

$$\frac{\partial U(\xi,t)}{\partial t}-c\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}\frac{\partial U(\xi,t)}{\partial \xi}+cU(\xi,t)\frac{\partial U(\xi,t)}{\partial \xi}-c^2\frac{\partial^2 U(\xi,t)}{\partial \xi^2}+\lambda U(\xi,t)=0. \tag{3}$$

In analogy with the conservative Burgers' case ($\lambda=0$), we look for solutions which obey the boundary conditions:

$$U(\xi,t) \qquad \frac{\partial U(\xi,t)}{\partial \xi} \qquad \frac{\partial^2 U(\xi,t)}{\partial \xi^2} \quad \to 0 \text{ as } \xi\to\infty. \tag{4}$$

Next, we introduce $Y=\tanh(\xi)$ as a new variable. As a consequence, $U(\xi,t)$ is replaced by $S(Y,t)$, $\partial/\partial\xi$ by $(1-Y^2)\ \partial/\partial Y$ and $\partial^2/\partial\xi^2$ by $(1-Y^2)\ [(1-Y^2)\ \partial/\partial Y]\ \partial/\partial Y$. Then (3) is rewritten as

$$c(1-Y^2)\left[\frac{\partial S(Y,t)}{\partial Y}\left(-\frac{\mathrm{d}\phi(t)}{\mathrm{d}t}+\frac{S(Y)}{2}+2cY\right)-c(1-Y^2)\frac{\partial^2 S(Y,t)}{\partial Y^2}\right]$$
$$+\frac{\partial S(Y,t)}{\partial t}+\lambda S(Y,t)=0. \tag{5}$$

As an ansatz, we postulate the following solution:

$$S(Y,t)=G(t)(1-Y)(1+a_1(t)Y+a_2(t)Y^2+a_3(t)Y^3+a_4(t)Y^4+\ldots)$$
$$\text{with } Y=\tanh[c(x-\phi(t))]. \tag{6}$$

It is based on the fact that for $\lambda=0$ (no dissipation), the exact solution is definitely written in terms of $S'(Y)$, or as $U'(\xi)$ (Y or ξ being the only variable involved). It reads:

$$S'(Y)=2c(1-Y) \qquad \text{with } Y=\tanh\xi=\tanh[c(x-vt)] \text{ and } v=2c. \tag{7}$$

The boundary condition here is $U'(\xi)\to 0$ for $\xi\to+\infty$ or $S'(Y)\to 0$ for $Y\to+1$. Hence, the boundary condition (4) likewise appears in (6).

This ansatz (6) is then substituted in equation (5). Next, one collects all terms proportional to Y^n ($n=0,1,2,3,\ldots$). This set will actually serve as a set of recurrence relations. Putting them equal to zero, we indeed get relations between the different time dependent unknown quantities $G(t)$, $\phi(t)$ and (t). We choose $a_1(t) = a_2(t) = 0$, so that in order Y^0 (lowest order) we arrive at:

$$cG(t)^2 - c\frac{d\phi(t)}{dt}G(t) - \lambda G(t) - \frac{dG(t)}{dt} = 0. \tag{8}$$

Obviously

$$\frac{d\phi(t)}{dt} = G(t) \tag{9}$$

and

$$\frac{dG(t)}{dt} = -\lambda G(t). \tag{10}$$

As a solution of these equations we choose:

$$G(t) = 2c\,e^{-\lambda t} \tag{11a}$$

and

$$\frac{d\phi(t)}{dt} = 2c\,e^{-\lambda t} \text{ or } \phi(t) = \frac{2c}{\lambda}(1 - e^{-\lambda t}). \tag{11b}$$

The initial condition for $G(t)$ at $t = 0$ is put equal to $2c$, in order to satisfy the earlier results for $\lambda = 0$. Indeed, in that case we have $G(t) = 2c$ and $\phi(t) = 2ct$. The amplitude, as well as the velocity, are now decaying functions in time.

In next order, i.e. all terms proportional to Y^1 we get the relation

$$2c^2G(t) + 6c^2G(t)a_3(t) - c\frac{d\phi(t)}{dt}G(t) = 0 \tag{12}$$

which yields

$$a_3(t) = \frac{1}{3}(e^{-\lambda t} - 1). \tag{13}$$

The fundamental quantities $a_4(t)$, $a_5(t)$, . . . can be found successively by investigating the associated recursion relation belonging to Y^4, Y^5, ... Amazingly, from the next recurrence relation, we get $a_4(t) = a_3(t)$. In general

$$a_{2n+2}(t) = a_{2n+1}(t) \text{ for } n = 1, 2, 3, \ldots. \tag{14}$$

The next relevant coefficient is:

$$a_5(t) = -\frac{1}{60c^2}(\lambda\,e^{-\lambda t} + 8c^2\,e^{-2\lambda t} - 40c^2\,e^{-\lambda t} + 32c^2). \tag{15}$$

Notice that $a_n(t) = 0$ $(n = 0, 1, 2, \ldots)$ for $\lambda = 0$.

Finally, the approximate solution of the damped Burgers equation reads:

$$u(x,t) = 2c\,e^{-\lambda t}(1 - Y)\{1 + a_3(t)Y^3(1 + Y) + a_5(t)Y^5(1 + Y) + a_7(t)Y^7(1 + Y)\ldots\}$$
$$\times \text{ with } Y = \tanh[c(x - (2c/\lambda)(1 - e^{-\lambda t}))]. \tag{16}$$

How many terms one has to take into account depends on the value of the damping factor λ. Remark, however, that for $Y \to 0$ and $Y \to -1$ (the boundary

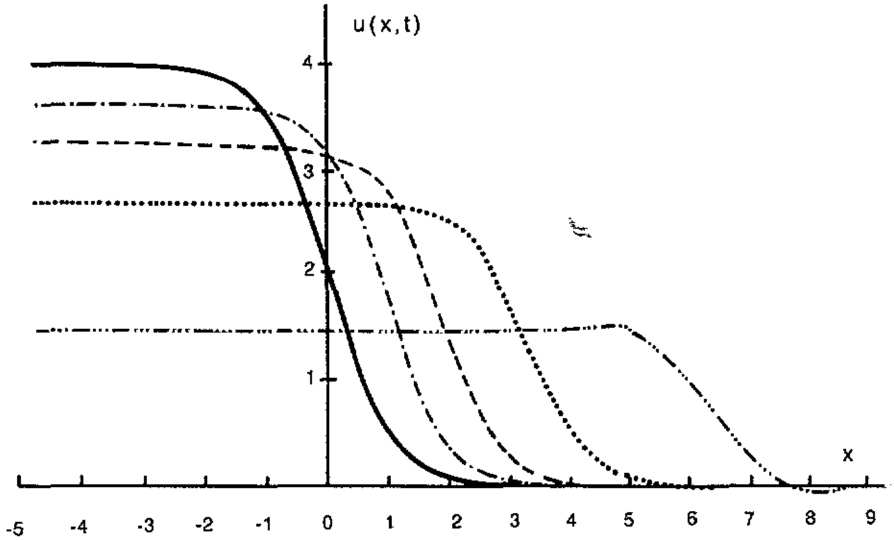

Figure 1. Dynamics of the damped shock wave. Parameters are $\lambda = 0.2$ and $c = 1$. Correction terms are included up to $O(Y^8)$. At $t = 0$ (full curve) the shock wave starts to damp. The next stages are: (*a*) $t = 0.5$ (chain curve), (*b*) $t = 1$ (broken curve), (*c*) $t = 2$ (dotted curve) and (*d*) $t = 5$ (dotted-chain curve). From $t \sim 5$ on, the tails start to show a bump-type of behaviour. This shock wave will eventually damp away at the point $x \sim 2c/\lambda = 10$.

condition for x or $\xi \to -\infty$) the solution is not affected by the correction terms between the last brackets. The boundary condition at the left-hand side ($Y \to -1$) just decays exponentially. This damped shock wave moves with a gradually diminishing velocity. As a result, it propagates a finite distance

$$L \sim 2c/\lambda. \tag{17}$$

The dynamical behaviour of this approximate solution is shown in figure 1. In this typical example, we observe that the shock wave structure lowers its amplitude considerably. As a consequence, the nonlinear term, responsible for steepening the wave, is less important and the steepening effect will be weakened. In the beginning the correction terms do not play a significant role and can in fact be ignored. After some time ($t \sim 5$) a bump-type of behaviour arises at the tails of the shock. Then the solution even becomes negative in the tail on the right-hand side. More correction terms must be taken into account, as is shown in figure 2. Note, however, that the wave in this stage is almost damped away ($t = 10$). The effect of these bumps is due to the correction terms $G_{2n+1}(Y) = (1-Y)Y^{2n+1}(1+Y)$, combined with the fact that for $t \to \infty$, the coefficients $a_{2n+1}(t)$ do not vanish. Indeed, in the limit $t \to \infty$, they behave as follows:

$$\begin{array}{lll} a_3(t) \to -(1/3) & a_5(t) \to -(8/15) & a_7(t) \to -(71/105) \\ a_9(t) \to -(248/315) & a_{11}(t) \to -(3043/3465), & \text{etc.} \ldots \end{array} \tag{18}$$

They represent a bounded series of terms because

$$\frac{a_5(\infty)}{a_3(\infty)} > \frac{a_7(\infty)}{a_5(\infty)} > \frac{a_9(\infty)}{a_7(\infty)} > \cdots \tag{19}$$

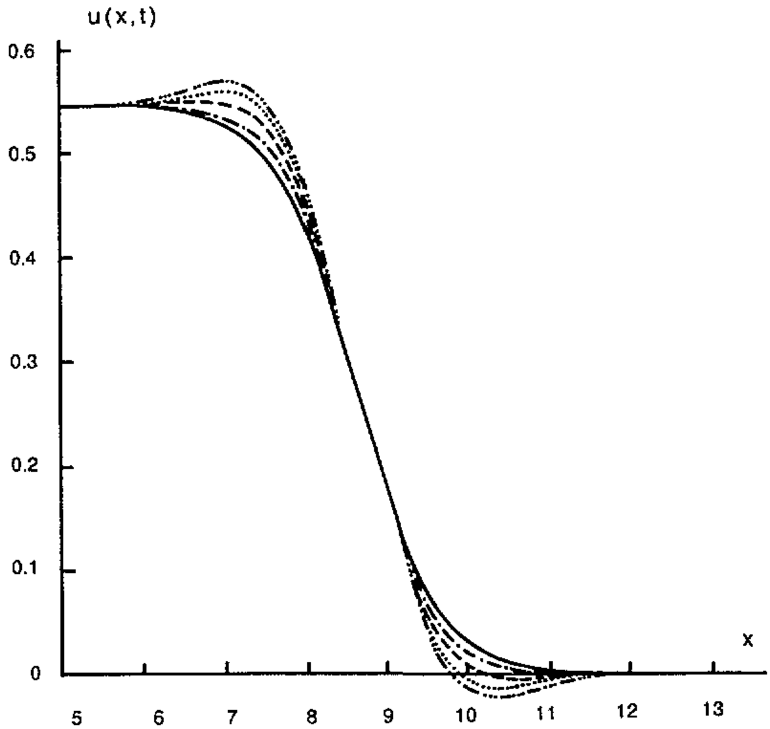

Figure 2. Influence of the correction terms at the tails of the approximate solution. The same approximative solution as in figure 1 is used, but sketched at a later time ($t=10$). The full curve represents the solution without correction terms ($a_i(t)=0$), and in the subsequent curves one more correction term (two orders in Y) is added respectively.

Convergence is assured because $G_{2n+1}(Y)$ vanishes for $n \to \infty$. Notice that in the case of a damped KdV equation a kind of negative bump also appears at the tail of the damped soliton (Leibovich 1979). As already mentioned, it is clear from figure 2 that the the main bulk of the shock ($Y \approx 0$) does not change much if one adds more correction terms.

With this example we have shown that it must be possible to use the tanh technique as a perturbation method to analyse nonlinear wave equations which are not exactly integrable. As a model, we have analysed the Burgers equation, in which dissipation occurs. To investigate other equations as well, there is one condition: the unperturbed solution must be a function (in one way or another) of a tanh. At present, more than 20 nonlinear wave and evolution equations possess this behaviour. The present analysis suggests that this technique can be advantageously used in more general cases.

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